An Algebraic Perspective on Duality in Logic and Language

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Examples of Duality

from logic:

- conjunction and disjunction
- $\forall$ and $\exists$ (first-order logic)
- $\Box$ and $\Diamond$ (modal logic)

from natural language:

- all of the above!
- 'already' and 'still' (S. Löbner)
- 'because' and 'although' (E. König)
- perhaps a semantic universal? (J. van Benthem)

Uniform Analysis

duality = **external negation** + **internal negation**

- $\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$
- $\forall x Px \equiv \neg\exists x \neg Px$
- **All** rooms are clean.
  - **No**, **some** rooms are **dirty** [not-clean]!
- John is **still** here.
  - **No**, he **already left** [not-here]!
- John is happy **because** he's rich.
  - **No**, he's happy **even though** he's **poor** [not-rich]!

What is Negation?

operation $neg_A : A \to A$ with two constraints:

- $neg_A \neq id_A$ — $neg_A(a) \neq a$ for at least some $a \in A$
- $neg_A \circ neg_A = id_A$ — $neg_A(neg_A(a)) = a$ for all $a \in A$

duality = non-trivial and symmetric phenomenon

☞ algebraic perspective: they form the group $\mathbb{Z}_2$

$\circ$	id_A	neg_A
id_A	id_A	neg_A
neg_A	neg_A	id_A

$\cong$

$\circ$	0	1
0	0	1
1	1	0

General Setup

n-ary operation $O : A^n \to B$

internal negation in all argument places:
$(O \circ neg_A)(a_1,\dots,a_n) := O(neg_A(a_1),\dots,neg_A(a_n))$

$I(O) = O$
$L(O) = neg_B \circ O$
$R(O) = O \circ neg_A$
$LR(O) = neg_B \circ O \circ neg_A = L(R(O)) = R(L(O))$

☞ these form the Klein 4-group V_4

$\circ$	I	L	R	LR
I	I	L	R	LR
L	L	I	LR	R
R	R	LR	I	L
LR	LR	R	L	I

$\cong$

Connecting Algebra and Syntax

well-known group-theoretical fact: the Klein 4-group V_4 is isomorphic to the direct product of $\mathbb{Z}_2$ with itself

$V_4 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$

duality = external negation + internal negation

$X(O) = (\ell, r)(O) = neg_B^{\ell} \circ O \circ neg_A^{r}$

$\circ$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

Applications and Examples

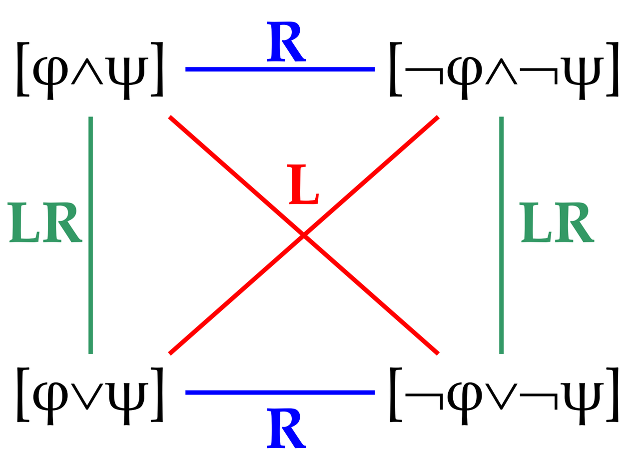
application: analysis of degenerate cases:

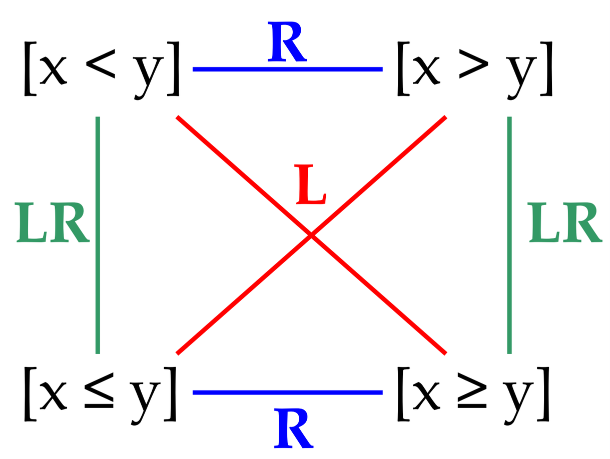
- operators that are their own dual
- operators that are their own internal negation

☞ duality group collapses into a single copy of $\mathbb{Z}_2$

example 1: $\wedge : Prop^2 \to Prop : ([\varphi],[\psi]) \mapsto [\varphi \wedge \psi]$
example 2: $< : \mathbb{R}^2 \to Prop : (x,y) \mapsto [x < y]$

neg_{Prop}: Prop→Prop: $[\varphi] \mapsto [\neg\varphi]$ $neg_{\mathbb{R}}$: $\mathbb{R} \to \mathbb{R}$: $x \mapsto -x$





Composed Operations

$O : A \to B$ and $O' : B \to C$ and so $O' \circ O : A \to C$ (unary: for simplicity)

duality group for $O' \circ O$: $V_4 \otimes V_4$? **no!**

componentwise: for $X,Y \in \{I, L, R, LR\}$ define $(X,Y)(O',O) := X(O') \circ Y(O)$

16 possibilities, but **pairwise collapsing**:

(I,I)	=	(R,L)	(L,I)	=	(LR,L)
(I,L)	=	(R,I)	(L,L)	=	(LR,I)
(I,R)	=	(R,LR)	(L,R)	=	(LR,LR)
(I,LR)	=	(R,R)	(L,LR)	=	(LR,R)

☞ these **8** form a group $G_8 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$

(0,0,0) (1,0,0)
(0,1,0) (1,1,0)
(0,0,1) (1,0,1)
(0,1,1) (1,1,1)

☞ syntactic perspective:

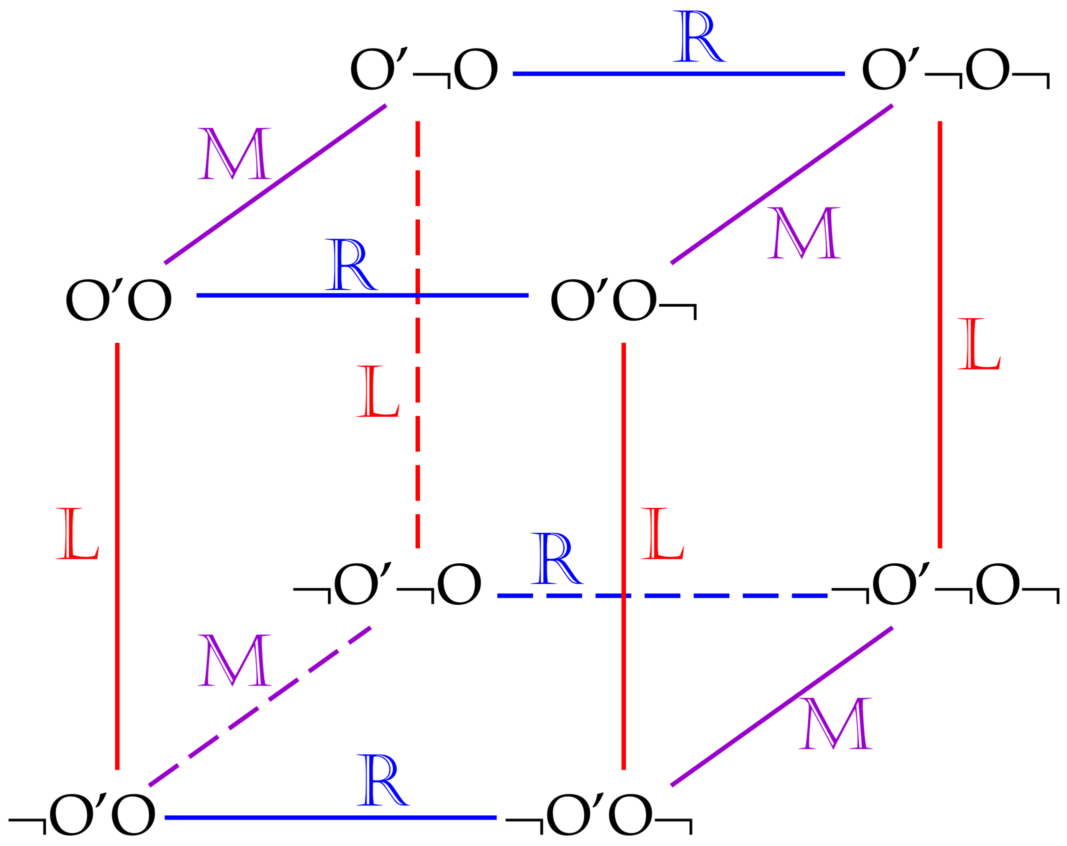
- **external negation**
- **internal negation**
- **intermediate negation**

Duality Cube

$X(O',O) = (\ell, m, r)(O',O) = neg_C^{\ell} \circ O' \circ neg_B^{m} \circ O \circ neg_A^{r}$

abbreviations: $I = (0,0,0)$ $L = (1,0,0)$ $M = (0,1,0)$ $R = (0,0,1)$

clarity: just write $\neg$ for neg_A, neg_B and neg_C



Internal Structure of the Duality Cube

duality cube ~ G_8 = {I, L, M, R, LM, LR, MR, LMR}

☞ use algebraic methods to study the duality cube

duality **squares in** the **cube** correspond **pairwise** with **sub**groups **of order 4** of G_8 (Lagrange)

{I, L, M, LM} {I, L, R, LR} {I, M, R, MR}	{I, L, MR, LMR} {I, M, LR, LMR} {I, R, LM, LMR}	{I, LM, LR, MR}
⇓	⇓	⇓
the 6 (3 × 2) outer faces	the 6 (3 × 2) diagonal planes	2 (1 × 2) 'twisted' squares
isometric embedding	affine embedding	topological embedding

ongoing research (with Hans Smessaert):

- 2 twisted squares ⇒ 2 (3-dimensional) tetrahedra
- 3 planar projections of tetrahedron ⇒ 3 squares

Further information: please contact me at lorenz.demey@hiw.kuleuven.be !

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